

65. (a) Since the line passes through the origin and has slope

$$\frac{1}{e}, \text{ its equation is } y = \frac{x}{e}.$$

- (b) The graph of $y = \ln x$ lies below the graph of the line

$$y = \frac{x}{e} \text{ for all positive } x \neq e. \text{ Therefore, } \ln x < \frac{x}{e} \text{ for all positive } x \neq e.$$

- (c) Multiplying by e , $e \ln x < x$ or $\ln x^e < x$.

- (d) Exponentiating both sides of $\ln x^e < x$, we have

$$e^{\ln x^e} < e^x, \text{ or } x^e < e^x \text{ for all positive } x \neq e.$$

- (e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

Quick Quiz Sections 3.7–3.9

1. E. $y = \frac{9}{2x} - \frac{x^2}{2}$

$$dy = -\frac{9}{2x^2} - x$$

$$dy = -\frac{9}{2(1)^2} - 1 = -\frac{11}{2}$$

2. A. $dy = \frac{d}{dx}(\cos^3(3x-2))$

$$dy = -9\cos^2(3x-2) \sin(3x-2)$$

3. C. $dy = \frac{d}{dx}(\sin^{-1}(2x))$

$$dy = \frac{2}{\sqrt{1-4x^2}}$$

4. (a) Differentiate implicitly:

$$\frac{d}{dx}(xy^2 - x^3y) = \frac{d}{dx}(6)$$

$$1 \cdot y^2 + x \cdot 2y \frac{dy}{dx} - \left(3x^2y + x^3 \frac{dy}{dx} \right) = 0$$

$$2xy \frac{dy}{dx} - x^3 \frac{dy}{dx} = 3x^2y - y^2$$

$$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$$

- (b) If $x = 1$, then $y^2 - y = 6$, so $y = -2$ or $y = 3$.

$$\text{at } (1, -2), \frac{dy}{dx} = \frac{3(1)^2(-2) - (-2)^2}{2(1)(-2) - (1)^3} = 2.$$

The tangent line is $y + 2 = 2(x - 1)$.

$$\text{At } (1, 3), \frac{dy}{dx} = \frac{3(1)^2(3) - 3^2}{2(1)(3) - 1^3} = 0.$$

The tangent line is $y = 3$.

- (c) The tangent line is vertical where $2xy - x^3 = 0$, which

implies $x = 0$ or $y = \frac{x^2}{2}$. There is no point on the curve

$$\text{where } x = 0. \text{ If } y = \frac{x^2}{2}, \text{ then } x \left(\frac{x^2}{2} \right) - x^3 \left(\frac{x^2}{2} \right) = 6.$$

Then the only solution to this equation is $x = \sqrt[3]{-24}$.

Chapter 3 Review Exercises

(pp. 181–184)

1. $\frac{dy}{dx} = \frac{d}{dx} \left(x^5 - \frac{1}{8}x^2 + \frac{1}{4}x \right) = 5x^4 - \frac{1}{4}x + \frac{1}{4}$

2. $\frac{dy}{dx} = \frac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

3. $\frac{dy}{dx} = \frac{d}{dx}(2 \sin x \cos x)$
 $= 2(\sin x) \frac{d}{dx}(\cos x) + 2(\cos x) \frac{d}{dx}(\sin x)$
 $= -2\sin^2 x + 2\cos^2 x$

Alternate solution:

$$\frac{dy}{dx} = \frac{d}{dx}(2 \sin x \cos x) = \frac{d}{dx} \sin 2x = (\cos 2x)(2)$$

$$= 2 \cos 2x$$

4. $\frac{dy}{dx} = \frac{d}{dx} \frac{2x+1}{2x-1} = \frac{(2x-1)(2) - (2x+1)(2)}{(2x-1)^2} = -\frac{4}{(2x-1)^2}$

5. $\frac{ds}{dt} = \frac{d}{dt} \cos(1-2t) = -\sin(1-2t)(-2) = 2 \sin(1-2t)$

6. $\frac{ds}{dt} = \frac{d}{dt} \cot\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right) \frac{d}{dt}\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right) \left(-\frac{2}{t^2}\right)$
 $= \frac{2}{t^2} \csc^2\left(\frac{2}{t}\right)$

7. $\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{x} + 1 + \frac{1}{\sqrt{x}} \right) = \frac{d}{dx} (x^{1/2} + 1 + x^{-1/2})$
 $= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

8. $\frac{dy}{dx} = \frac{d}{dx}(x\sqrt{2x+1}) = (x) \left(\frac{1}{2\sqrt{2x+1}} \right) (2) + (\sqrt{2x+1})(1)$
 $= \frac{x + (2x+1)}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$

9. $\frac{dr}{d\theta} = \frac{d}{d\theta} \sec(1+3\theta) = \sec(1+3\theta) \tan(1+3\theta)(3)$
 $= 3 \sec(1+3\theta) \tan(1+3\theta)$

10. $\frac{dr}{d\theta} = \frac{d}{d\theta} \tan^2(3-\theta^2)$
 $= 2 \tan(3-\theta^2) \frac{d}{d\theta} \tan(3-\theta^2)$
 $= 2 \tan(3-\theta^2) \sec^2(3-\theta^2)(-2\theta)$
 $= -4\theta \tan(3-\theta^2) \sec^2(3-\theta^2)$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^2 \csc 5x)$
 $= (x^2)(-\csc 5x \cot 5x)(5) + (\csc 5x)(2x)$
 $= -5x^2 \csc 5x \cot 5x + 2x \csc 5x$

$$12. \frac{dy}{dx} = \frac{d}{dx} \ln \sqrt{x} = \frac{1}{\sqrt{x}} \frac{d}{dx} \sqrt{x} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}, x > 0$$

$$13. \frac{dy}{dx} = \frac{d}{dx} \ln(1+e^x) = \frac{1}{1+e^x} \frac{d}{dx} (1+e^x) = \frac{e^x}{1+e^x}$$

$$14. \frac{dy}{dx} = \frac{d}{dx} (xe^{-x}) = (x)(e^{-x})(-1) + (e^{-x})(1) = -xe^{-x} + e^{-x}$$

$$15. \frac{dy}{dx} = \frac{d}{dx} (e^{1+\ln x}) = \frac{d}{dx} (e^1 e^{\ln x}) = \frac{d}{dx} (ex) = e$$

$$16. \frac{dy}{dx} = \frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx} (\sin x) = \frac{\cos x}{\sin x} = \cot x, \text{ for values of } x \text{ in the intervals } (k\pi, (k+1)\pi), \text{ where } k \text{ is even.}$$

$$17. \frac{dr}{dx} = \frac{d}{dx} \ln(\cos^{-1} x) = \frac{1}{\cos^{-1} x} \frac{d}{dx} \cos^{-1} x \\ = \frac{1}{\cos^{-1} x} \left(-\frac{1}{\sqrt{1-x^2}} \right) = -\frac{1}{\cos^{-1} x \sqrt{1-x^2}}$$

$$18. \frac{dr}{d\theta} = \frac{d}{d\theta} \log_2(\theta^2) = \frac{1}{\theta^2 \ln 2} \frac{d}{d\theta} (\theta^2) = \frac{2\theta}{\theta^2 \ln 2} = \frac{2}{\theta \ln 2}$$

$$19. \frac{ds}{dt} = \frac{d}{dt} \log_5(t-7) = \frac{1}{(t-7) \ln 5} \frac{d}{dt} (t-7) = \frac{1}{(t-7) \ln 5}, t > 7$$

$$20. \frac{ds}{dt} = \frac{d}{dt} (8^{-t}) = 8^{-t} (\ln 8) \frac{d}{dt} (-t) = -8^{-t} \ln 8$$

21. Use logarithmic differentiation.

$$y = x^{\ln x} \\ \ln y = \ln(x^{\ln x}) \\ \ln y = (\ln x)(\ln x) \\ \frac{d}{dx} \ln y = \frac{d}{dx} (\ln x)^2 \\ \frac{1}{y} \frac{dy}{dx} = 2 \ln x \frac{d}{dx} \ln x \\ \frac{dy}{dx} = \frac{2y \ln x}{x} \\ \frac{dy}{dx} = \frac{2x^{\ln x} \ln x}{x}$$

$$22. \frac{dy}{dx} = \frac{d}{dx} \frac{(2x)2^x}{\sqrt{x^2+1}} \\ = \frac{\sqrt{x^2+1} \frac{d}{dx} [(2x)2^x] - (2x)(2^x) \frac{d}{dx} \sqrt{x^2+1}}{x^2+1} \\ = \frac{\sqrt{x^2+1} [(2x)(2^x)(\ln 2) + (2^x)(2)] - (2x)(2^x) \frac{1}{2\sqrt{x^2+1}} (2x)}{x^2+1}$$

$$= \frac{(x^2+1)(2^x)(2x \ln 2 + 2) - 2x^2(2^x)}{(x^2+1)^{3/2}} \\ = \frac{(2 \cdot 2^x)[(x^2+1)(x \ln 2 + 1) - x^2]}{(x^2+1)^{3/2}} \\ = \frac{(2 \cdot 2^x)(x^3 \ln 2 + x^2 + x \ln 2 + 1 - x^2)}{(x^2+1)^{3/2}} \\ = \frac{(2 \cdot 2^x)(x^3 \ln 2 + x \ln 2 + 1)}{(x^2+1)^{3/2}}$$

Alternate solution, using logarithmic differentiation:

$$y = \frac{(2x)2^x}{\sqrt{x^2+1}} \\ \ln y = (2x) + \ln(2^x) - \ln \sqrt{x^2+1} \\ \ln y = \ln 2 + \ln x + x \ln 2 - \frac{1}{2} \ln(x^2+1) \\ \frac{d}{dx} \ln y = \frac{d}{dx} [\ln 2 + \ln x + \ln 2 - \frac{1}{2} \ln(x^2+1)] \\ \frac{1}{y} \frac{dy}{dx} = 0 + \frac{1}{x} + \ln 2 - \frac{1}{2} \frac{1}{x^2+1} (2x) \\ \frac{dy}{dx} = y \left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1} \right) \\ \frac{dy}{dx} = \frac{(2x)2^x}{\sqrt{x^2+1}} \left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1} \right)$$

$$23. \frac{dy}{dx} = \frac{d}{dx} e^{\tan^{-1} x} = e^{\tan^{-1} x} \frac{d}{dx} \tan^{-1} x = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$24. \frac{dy}{du} = \frac{d}{dx} \sin^{-1} \sqrt{1-u^2} \\ = \frac{1}{\sqrt{1-(\sqrt{1-u^2})^2}} \frac{d}{du} \sqrt{1-u^2} \\ = \frac{1}{\sqrt{u^2}} \frac{1}{2\sqrt{1-u^2}} (-2u) = \frac{u}{|u|\sqrt{1-u^2}}$$

$$25. \frac{dy}{dt} = \frac{d}{dt} \left(t \sec^{-1} t - \frac{1}{2} \ln t \right) \\ = (t) \left(\frac{1}{|t|\sqrt{t^2-1}} \right) + (\sec^{-1} t)(1) - \frac{1}{2t} \\ = \frac{t}{|t|\sqrt{t^2-1}} + \sec^{-1} t - \frac{1}{2t}$$

$$26. \frac{dy}{dt} = \frac{d}{dt} [(1+t^2) \cot^{-1} 2t] \\ = (1+t^2) \left(-\frac{1}{1+(2t)^2} \right) (2) + (\cot^{-1} 2t) (2t) \\ = -\frac{2+2t^2}{1+4t^2} + 2t \cot^{-1} 2t$$

$$\begin{aligned}
 27. \frac{dy}{dz} &= \frac{d}{dz}(z \cos^{-1} z - \sqrt{1-z^2}) \\
 &= (z) \left(-\frac{1}{\sqrt{1-z^2}} \right) + (\cos^{-1} z)(1) - \frac{1}{2\sqrt{1-z^2}}(-2z) \\
 &= -\frac{z}{\sqrt{1-z^2}} + \cos^{-1} z + \frac{z}{\sqrt{1-z^2}} \\
 &= \cos^{-1} z
 \end{aligned}$$

$$\begin{aligned}
 28. \frac{dy}{dx} &= \frac{d}{dx}(2\sqrt{x-1} \csc^{-1} \sqrt{x}) \\
 &= (2\sqrt{x-1}) \left(-\frac{1}{\sqrt{x}(\sqrt{x})^2-1} \right) \left(\frac{1}{2\sqrt{x}} \right) \\
 &\quad + (2 \csc^{-1} \sqrt{x}) \left(\frac{1}{2\sqrt{x-1}} \right) \\
 &= -\frac{\sqrt{x-1}}{(\sqrt{x})^2 \sqrt{x-1}} + \frac{\csc^{-1} \sqrt{x}}{\sqrt{x-1}} \\
 &= -\frac{1}{x} + \frac{\csc^{-1} \sqrt{x}}{\sqrt{x-1}}
 \end{aligned}$$

$$\begin{aligned}
 29. \frac{dy}{dx} &= \frac{d}{dx} \csc^{-1}(\sec x) \\
 &= \left(-\frac{1}{|\sec x| \sqrt{\sec^2 x - 1}} \right) \frac{d}{dx}(\sec x) \\
 &= -\frac{1}{|\sec x| \sqrt{\tan^2 x - 1}} \sec x \tan x \\
 &= -\frac{\sec x \tan x}{|\sec x \tan x|} \\
 &= -\frac{1 \sin x}{\cos x \cos x} = -\frac{\sin x}{|\sin x|} \\
 &= \begin{cases} -1, & 0 \leq x < \pi, \quad x \neq \frac{\pi}{2} \\ 1, & \pi < x \leq 2\pi, \quad x \neq \frac{3\pi}{2} \end{cases}
 \end{aligned}$$

Alternate method:

On the domain $0 \leq x \leq 2\pi$, $x \neq \frac{\pi}{2}$, $x \neq \frac{3\pi}{2}$, we may

rewrite the function as follows:

$$\begin{aligned}
 y &= \csc^{-1}(\sec x) \\
 &= \frac{\pi}{2} - \sec^{-1}(\sec x) \\
 &= \frac{\pi}{2} - \cos^{-1}(\cos x) \\
 &= \begin{cases} \frac{\pi}{2} - x, & 0 \leq x \leq \pi, \quad x \neq \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi - x), & \pi < x \leq 2\pi, \quad x \neq \frac{3\pi}{2} \end{cases} \\
 &= \begin{cases} \frac{\pi}{2} - x, & 0 \leq x \leq \pi, \quad x \neq \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi + x), & \pi < x \leq 2\pi, \quad x \neq \frac{3\pi}{2} \end{cases}
 \end{aligned}$$

$$\text{Therefore, } \frac{dy}{dx} = \begin{cases} -1, & 0 \leq x < \pi, \quad x \neq \frac{\pi}{2} \\ 1, & \pi < x \leq 2\pi, \quad x \neq \frac{3\pi}{2} \end{cases}$$

Note that the derivative exists at 0 and 2π only because these are the endpoints of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at these points.

$$\begin{aligned}
 30. \frac{dr}{d\theta} &= \frac{d}{d\theta} \left(\frac{1+\sin\theta}{1-\cos\theta} \right)^2 \\
 &= 2 \left(\frac{1+\sin\theta}{1-\cos\theta} \right) \left(\frac{(1-\cos\theta)(\cos\theta) - (1+\sin\theta)(\sin\theta)}{(1-\cos\theta)^2} \right) \\
 &= 2 \left(\frac{1+\sin\theta}{1-\cos\theta} \right) \left(\frac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1-\cos\theta)^2} \right) \\
 &= 2 \left(\frac{1+\sin\theta}{1-\cos\theta} \right) \left(\frac{\cos\theta - \sin\theta - 1}{(1-\cos\theta)^2} \right)
 \end{aligned}$$

31. Since $y = \ln x^2$ is defined for all

$x \neq 0$ and $\frac{dy}{dx} = \frac{1}{x^2} \frac{d}{dx}(x^2) = \frac{2x}{x^2} = \frac{2}{x}$, the function is differentiable for all $x \neq 0$.

32. Since $y = \sin x - x \cos x$ is defined for all real x and

$\frac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x \sin x$, the function is differentiable for all real x .

33. Since $y = \sqrt{\frac{1-x}{1+x^2}}$ is defined for all $x < 1$ and

$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1-x}{1+x^2}}} \frac{(1+x^2)(-1) - (1-x)(2x)}{(1+x^2)^2}$
 $= \frac{x^2 - 2x - 1}{2\sqrt{1-x}(1+x^2)^{3/2}}$, which is defined only for $x < 1$,
the function is differentiable for all $x < 1$.

34. Since $y = (2x-7)^{-1}(x+5) = \frac{x+5}{2x-7}$ is defined for all

$x \neq \frac{7}{2}$ and $\frac{dy}{dx} = \frac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2} = -\frac{17}{(2x-7)^2}$,
the function is differentiable for all $x \neq \frac{7}{2}$.

35. Use implicit differentiation.

$$\begin{aligned}
 xy + 2x + 3y &= 1 \\
 \frac{dy}{dx}(xy) + \frac{d}{dx}(2x) + \frac{d}{dx}(3y) &= \frac{d}{dx}(1) \\
 x \frac{dy}{dx} + (y)(1) + 2 + 3 \frac{dy}{dx} &= 0 \\
 (x+3) \frac{dy}{dx} &= -(y+2) \\
 \frac{dy}{dx} &= -\frac{y+2}{x+3}
 \end{aligned}$$

36. Use implicit differentiation.

$$\begin{aligned}
 5x^{4/5} + 10y^{6/5} &= 15 \\
 \frac{d}{dx}(5x^{4/5}) + \frac{d}{dx}(10y^{6/5}) &= \frac{d}{dx}(15) \\
 4x^{-1/5} + 12y^{1/5} \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= -\frac{4x^{-1/5}}{12y^{1/5}} = -\frac{1}{3(xy)^{1/5}}
 \end{aligned}$$

37. Using implicit differentiation.

$$\begin{aligned}
 \sqrt{xy} &= 1 \\
 \frac{d}{dx}\sqrt{xy} &= \frac{d}{dx}(1) \\
 \frac{1}{2\sqrt{xy}} \left[x \frac{dy}{dx} + (y)(1) \right] &= 0 \\
 x \frac{dy}{dx} + y &= 0 \\
 \frac{dy}{dx} &= -\frac{y}{x}
 \end{aligned}$$

Alternate method:

38. Use implicit differentiation.

$$\begin{aligned}
 y^2 &= \frac{x}{x+1} \\
 \frac{d}{dx}y^2 &= \frac{d}{dx} \frac{x}{x+1} \\
 2y \frac{dy}{dx} &= \frac{(x+1)(1) - (x)(1)}{(x+1)^2} \\
 \frac{dy}{dx} &= \frac{1}{2y(x+1)^2}
 \end{aligned}$$

39. $x^3 + y^3 = 1$

$$\begin{aligned}
 \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) &= \frac{d}{dx}(1) \\
 3x^2 + 3y^2 y' &= 0 \\
 y' &= -\frac{x^2}{y^2} \\
 y'' &= \frac{d}{dx} \left(-\frac{x^2}{y^2} \right) \\
 &= -\frac{(y^2)(2x) - (x^2)(2y)(y')}{y^4} \\
 &= -\frac{(y^2)(2x) - (x^2)(2y) \left(-\frac{x^2}{y^2} \right)}{y^4} \\
 &= -\frac{2xy^3 + 2x^4}{y^5} \\
 &= -\frac{2x(x^3 + y^3)}{y^5} \\
 &= -\frac{2x}{y^5}
 \end{aligned}$$

since $x^3 + y^3 = 1$

40. $y^2 = 1 - \frac{2}{x}$

$$\begin{aligned}
 \frac{d}{dx}(y^2) &= \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{2}{x}\right) \\
 2yy' &= \frac{2}{x^2} \\
 y' &= \frac{2}{x^2(2y)} = \frac{1}{x^2 y} \\
 y'' &= \frac{d}{dx} \left(\frac{1}{x^2 y} \right) \\
 &= -\frac{1}{(x^2 y)^2} \frac{d}{dx}(x^2 y) \\
 &= -\frac{1}{(x^2 y)^2} [(x^2)(y') + (y)(2x)] \\
 &= -\frac{1}{(x^2 y)^2} \left[(x^2) \left(\frac{1}{x^2 y} \right) + 2xy \right] \\
 &= -\frac{1}{x^4 y^2} \left(\frac{1}{y} + 2xy \right) \\
 &= -\frac{1 + 2xy^2}{x^4 y^3}
 \end{aligned}$$

41. $y^3 + y = 2 \cos x$

$$\begin{aligned}
 \frac{d}{dx}(y^3) + \frac{d}{dx}(y) &= \frac{d}{dx}(2 \cos x) \\
 3y^2 y' + y' &= -2 \sin x \\
 (3y^2 + 1)y' &= -2 \sin x \\
 y' &= -\frac{2 \sin x}{3y^2 + 1} \\
 y'' &= \frac{d}{dx} \left(-\frac{2 \sin x}{3y^2 + 1} \right) \\
 &= -\frac{(3y^2 + 1)(2 \cos x) - (2 \sin x)(6yy')}{(3y^2 + 1)^2} \\
 &= -\frac{(3y^2 + 1)(2 \cos x) - (12y \sin x) \left(-\frac{2 \sin x}{3y^2 + 1} \right)}{(3y^2 + 1)^2} \\
 &= -2 \frac{(3y^2 + 1)^2 \cos x + 12y \sin^2 x}{(3y^2 + 1)^3}
 \end{aligned}$$

42. $x^{1/3} + y^{1/3} = 4$

$$\frac{d}{dx}(x^{1/3}) + \frac{d}{dx}(y^{1/3}) = \frac{d}{dx}(4)$$

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$$

$$y' = -\frac{x^{-2/3}}{y^{-2/3}} = -\left(\frac{y}{x}\right)^{2/3}$$

$$\begin{aligned} y'' &= \frac{d}{dx} \left[-\left(\frac{y}{x}\right)^{2/3} \right] \\ &= -\frac{2}{3} \left(\frac{y}{x}\right)^{-1/3} \left(\frac{xy' - (y)(1)}{x^2} \right) \\ &= -\frac{2}{3} \left(\frac{y}{x}\right)^{-1/3} \left(\frac{(x) \left[-\left(\frac{y}{x}\right)^{2/3} \right] - y}{x^2} \right) \\ &= -\frac{2}{3} x^{1/3} y^{-1/3} (-x^{-5/3} y^{2/3} - x^{-2} y) \\ &= \frac{2}{3} x^{-4/3} y^{1/3} + \frac{2}{3} x^{-5/3} y^{2/3} \end{aligned}$$

43. $y' = 2x^3 - 3x - 1$,

$$y'' = 6x^2 - 3,$$

$$y''' = 12x,$$

$$y^{(4)} = 12, \text{ and the rest are all zero.}$$

44. $y' = \frac{x^4}{24}$,

$$y'' = \frac{x^3}{6},$$

$$6y''' = \frac{x^2}{2},$$

$$y^{(4)} = x,$$

$$y^{(5)} = 1, \text{ and the rest are all zero.}$$

45. $\frac{dy}{dx} = \frac{d}{dx} \sqrt{x^2 - 2x} = \frac{1}{2\sqrt{x^2 - 2x}} (2x - 2) = \frac{x-1}{\sqrt{x^2 - 2x}}$

$$\text{At } x = 3, \text{ we have } y = \sqrt{3^2 - 2(3)} = \sqrt{3}$$

$$\text{and } \frac{dy}{dx} = \frac{3-1}{\sqrt{3^2 - 2(3)}} = \frac{2}{\sqrt{3}}.$$

(a) Tangent: $y = \frac{2}{\sqrt{3}}(x-3) + \sqrt{3}$ or $y = \frac{2}{\sqrt{3}}x - \sqrt{3}$

(b) Normal: $y = -\frac{\sqrt{3}}{2}(x-3) + \sqrt{3}$ or $y = -\frac{\sqrt{3}}{2}x + \frac{5\sqrt{3}}{2}$

46. $\frac{dy}{dx} = \frac{d}{dx}(4 + \cot x - 2 \csc x)$
 $= -\csc^2 x + 2 \csc x \cot x$

$$\text{At } x = \frac{\pi}{2}, \text{ we have}$$

$$y = 4 + \cot \frac{\pi}{2} - 2 \csc \frac{\pi}{2} = 4 + 0 - 2 = 2 \text{ and}$$

$$\frac{dy}{dx} = -\csc^2 \frac{\pi}{2} + 2 \csc \frac{\pi}{2} \cot \frac{\pi}{2} = -1 + 2(1)(0) = -1.$$

(a) Tangent: $y = -1\left(x - \frac{\pi}{2}\right) + 2$ or $y = -x + \frac{\pi}{2} + 2$

(b) Normal: $y = -1\left(x - \frac{\pi}{2}\right) + 2$ or $y = x - \frac{\pi}{2} + 2$

47. Use implicit differentiation.

$$x^2 + 2y^2 = 9$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(9)$$

$$2x + 4y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$$

$$\text{Slope at } (1, 2): -\frac{1}{2(2)} = -\frac{1}{4}$$

(a) Tangent: $y = -\frac{1}{4}(x-1) + 2$ or $y = -\frac{1}{4}x + \frac{9}{4}$

(b) Normal: $y = 4(x-1) + 2$ or $y = 4x - 2$

48. Use implicit differentiation.

$$x + \sqrt{xy} = 6$$

$$\frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{xy}) = \frac{d}{dx}(6)$$

$$1 + \frac{1}{2\sqrt{xy}} \left[(x) \left(\frac{dy}{dx} \right) + (y)(1) \right] = 0$$

$$\frac{x}{2\sqrt{xy}} \frac{dy}{dx} = -1 - \frac{y}{2\sqrt{xy}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2\sqrt{xy}}{x} \left(-1 - \frac{y}{2\sqrt{xy}} \right) \\ &= -2\sqrt{\frac{y}{x}} - \frac{y}{x} \end{aligned}$$

$$\text{Slope at } (4, 1): -2\sqrt{\frac{1}{4}} - \frac{1}{4} = -\frac{2}{2} - \frac{1}{4} = -\frac{5}{4}$$

(a) Tangent: $y = -\frac{5}{4}(x-4) + 1$ or $y = -\frac{5}{4}x + 6$

(b) Normal: $y = -\frac{4}{5}(x-4) + 1$ or $y = -\frac{4}{5}x + \frac{11}{5}$

$$49. \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2 \sin t}{2 \cos t} = -\tan t$$

$$\text{At } t = \frac{3\pi}{4}, \text{ we have } x = 2 \sin \frac{3\pi}{4} = \sqrt{2},$$

$$y = 2 \cos \frac{3\pi}{4} = -\sqrt{2}, \text{ and } \frac{dy}{dx} = -\tan \frac{3\pi}{4} = 1.$$

The equation of the tangent line is

$$y = 1(x - \sqrt{2}) + (-\sqrt{2}), \text{ or } y = x - 2\sqrt{2}.$$

$$50. \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 \cos t}{-3 \sin t} = -\frac{4}{3} \cot t$$

$$\text{At } t = \frac{3\pi}{4}, \text{ we have } x = 3 \cos \frac{3\pi}{4} = -\frac{3\sqrt{2}}{2},$$

$$y = 4 \sin \frac{3\pi}{4} = 2\sqrt{2}, \text{ and } \frac{dy}{dx} = -\frac{4}{3} \cot \frac{3\pi}{4} = \frac{4}{3}.$$

The equation of the tangent line is

$$y = \frac{4}{3} \left(x + \frac{3\sqrt{2}}{2} \right) + 2\sqrt{2}, \text{ or } y = \frac{4}{3}x + 4\sqrt{2}.$$

$$51. \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5 \sec^2 t}{3 \sec t \tan t} = \frac{5 \sec t}{3 \tan t} = \frac{5}{3 \sin t}$$

$$\text{At } t = \frac{\pi}{6}, \text{ we have } x = 3 \sec \frac{\pi}{6} = 2\sqrt{3},$$

$$y = 5 \tan \frac{\pi}{6} = \frac{5\sqrt{3}}{3}, \text{ and } \frac{dy}{dx} = \frac{5}{3 \sin \left(\frac{\pi}{6} \right)} = \frac{10}{3}.$$

The equation of the tangent line is

$$y = \frac{10}{3}(x - 2\sqrt{3}) + \frac{5\sqrt{3}}{3}, \text{ or } y = \frac{10}{3}x - 5\sqrt{3}.$$

$$52. \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \cos t}{-\sin t}$$

$$\text{At } t = -\frac{\pi}{4}, \text{ we have } x = \cos \left(-\frac{\pi}{4} \right) = \frac{\sqrt{2}}{2},$$

$$y = -\frac{\pi}{4} + \sin \left(-\frac{\pi}{4} \right) = -\frac{\pi}{4} - \frac{\sqrt{2}}{2}, \text{ and}$$

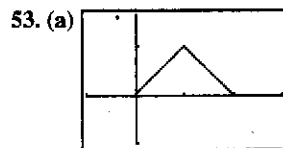
$$\frac{dy}{dx} = \frac{1 + \cos \left(-\frac{\pi}{4} \right)}{-\sin \left(-\frac{\pi}{4} \right)} = \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2} + 1.$$

The equation of the tangent line is

$$y = (\sqrt{2} + 1) \left(x - \frac{\sqrt{2}}{2} \right) - \frac{\pi}{4} - \frac{\sqrt{2}}{2}, \text{ or}$$

$$y = (1 + \sqrt{2})x - \sqrt{2} - 1 - \frac{\pi}{4}.$$

This is approximately $y = 2.414x - 3.200$.



$[-1, 3]$ by $[-1, 5/3]$

(b) Yes, because both of the one-sided limits as $x \rightarrow 1$ are equal to $f(1) = 1$.

(c) No, because the left-hand derivative at $x = 1$ is $+1$ and the right-hand derivative at $x = 1$ is -1 .

54. (a) The function is continuous for all values of m , because the right-hand limit as $x \rightarrow 0$ is equal to $f(0) = 0$ for any value of m .

(b) The left-hand derivative at $x = 0$ is $2\cos(2 \cdot 0) = 2$, and the right-hand derivative at $x = 0$ is m , so in order for the function to be differentiable at $x = 0$, m must be 2.

55. (a) For all $x \neq 0$

(b) At $x = 0$

(c) Nowhere

56. (a) For all x

(b) Nowhere

(c) Nowhere

57. Note that $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (2x - 3) = -3$ and

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x - 3) = -3$. Since these values agree with $f(0)$, the function is continuous at $x = 0$. On the other hand,

$f'(x) = \begin{cases} 2, & -1 \leq x < 0 \\ 1, & 0 < x \leq 4 \end{cases}$, so the derivative is undefined at $x = 0$.

(a) $[-1, 0) \cup (0, 4]$

(b) At $x = 0$

(c) Nowhere in its domain

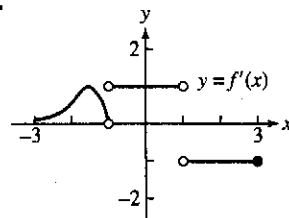
58. Note that the function is undefined at $x = 0$.

(a) $[-2, 0) \cup (0, 2]$

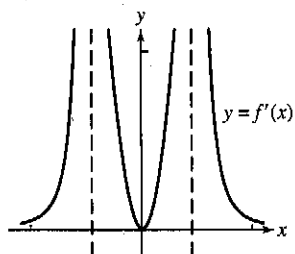
(b) Nowhere

(c) Nowhere in its domain

59.



60.

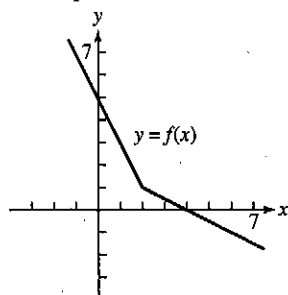


61. (a) iii

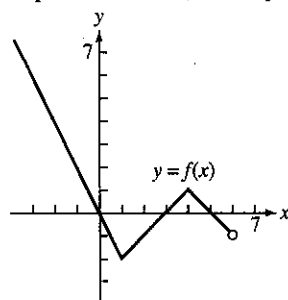
(b) i

(c) ii

62. The graph passes through (0, 5) and has slope -2 for $x < 2$ and slope -0.5 for $x > 2$.



63. The graph passes through $(-1, 2)$ and has slope -2 for $x < 1$, slope 1 for $1 < x < 4$, and slope -1 for $4 < x < 6$.



64. i. If $f(x) = \frac{9}{28}x^{7/3} + 9$, then $f'(x) = \frac{3}{4}x^{4/3}$ and

$f''(x) = x^{1/3}$, which matches the given equation.

ii. If $f'(x) = \frac{9}{28}x^{7/3} - 2$, then $f''(x) = \frac{3}{4}x^{4/3}$, which contradicts the given equation $f''(x) = x^{1/3}$.

iii. If $f'(x) = \frac{3}{4}x^{4/3} + 6$, then $f''(x) = x^{1/3}$, which matches the given equation.

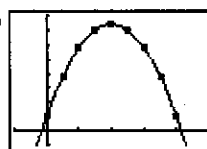
iv. If $f(x) = \frac{3}{4}x^{4/3} - 4$, then $f'(x) = x^{1/3}$ and

$f''(x) = \frac{1}{3}x^{-2/3}$, which contradicts the given equation

$f''(x) = x^{1/3}$.

Answer is D: i and iii only could be true. Note, however that i and iii could not simultaneously be true.

65. (a)



$[-1, 5]$ by $[-10, 80]$

(b) t interval

avg. vel.

$$[0, 0.5] \quad \frac{38 - 10}{0.5 - 0} = 56$$

$$[0.5, 1] \quad \frac{58 - 38}{1 - 0.5} = 40$$

$$[1, 1.5] \quad \frac{70 - 58}{1.5 - 1} = 24$$

$$[1.5, 2] \quad \frac{74 - 70}{2 - 1.5} = 8$$

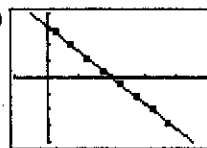
$$[2, 2.5] \quad \frac{70 - 74}{2.5 - 2} = -8$$

$$[2.5, 3] \quad \frac{58 - 70}{3 - 2.5} = -24$$

$$[3, 3.5] \quad \frac{38 - 58}{3.5 - 3} = -40$$

$$[3.5, 4] \quad \frac{10 - 38}{4 - 3.5} = -56$$

(c)



$[-1, 5]$ by $[-80, 80]$

(d) Average velocity is a good approximation to velocity.

66. (a) $\frac{d}{dx} [\sqrt{x} f(x)] = \sqrt{x} f'(x) + \frac{1}{2\sqrt{x}} f(x)$

At $x = 1$, the derivative is

$$\sqrt{1} f'(1) + \frac{1}{2\sqrt{1}} f(1) = 1 \left(\frac{1}{5} \right) + \left(\frac{1}{2} \right) (-3) = -\frac{13}{10}.$$

(b) $\frac{d}{dx} \sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

At $x = 0$, the derivative is $\frac{f'(0)}{2\sqrt{f(0)}} = -\frac{2}{2\sqrt{9}} = -\frac{1}{3}.$

(c) $\frac{d}{dx} f(\sqrt{x}) = f'(\sqrt{x}) \frac{d}{dx} \sqrt{x} = \frac{f'(\sqrt{x})}{2\sqrt{x}}$

At $x = 1$, the derivative is $\frac{f'(\sqrt{1})}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{\frac{1}{5}}{2} = \frac{1}{10}.$

(d) $\frac{d}{dx} f(1 - 5 \tan x) = f'(1 - 5 \tan x) (-5 \sec^2 x)$

At $x = 0$, the derivative is

$$f'(1 - 5 \tan 0) (-5 \sec^2 0) = f'(1) (-5) = \left(\frac{1}{5} \right) (-5) = -1.$$

66. Continued

$$(e) \frac{d}{dx} \frac{f(x)}{2 + \cos x} = \frac{(2 + \cos x)(f'(x)) - (f(x))(-\sin x)}{(2 + \cos x)^2}$$

At $x = 0$, the derivative is

$$\frac{(2 + \cos 0)(f'(0)) - (f(0))(-\sin 0)}{(2 + \cos 0)^2} = \frac{3f'(0)}{3^2} = \frac{2}{3}$$

$$(f) \frac{d}{dx} \left[10 \sin \left(\frac{\pi x}{2} \right) f^2(x) \right]$$

$$= 10 \left(\sin \frac{\pi x}{2} \right) (2f(x)f'(x)) + 10f^2(x) \left(\cos \frac{\pi x}{2} \right) \left(\frac{\pi}{2} \right)$$

$$= 20f(x)f'(x) \sin \frac{\pi x}{2} + 5\pi f^2(x) \cos \frac{\pi x}{2}$$

At $x = 1$, the derivative is

$$20f(1)f'(1) \sin \frac{\pi}{2} + 5\pi f^2(1) \cos \frac{\pi}{2}$$

$$= 20(-3) \left(\frac{1}{5} \right) (1) + 5\pi(-3)^2(0)$$

$$= -12.$$

$$67. (a) \frac{d}{dx} [3f(x) - g(x)] = 3f'(x) - g'(x)$$

At $x = -1$, the derivative is

$$3f'(-1) - g'(-1) = 3(2) - 1 = 5.$$

$$(b) \frac{d}{dx} [f^2(x)g^3(x)]$$

$$= f^2(x) \cdot 3g^2(x)g'(x) + g^3(x) \cdot 2f(x)f'(x)$$

$$= f(x)g^2(x) [3f(x)g'(x) + 2g(x)f'(x)]$$

At $x = 0$, the derivative is

$$f(0)g^2(0) [3f(0)g'(0) + 2g(0)f'(0)]$$

$$= (-1)(-3)^2 [3(-1)(4) + 2(-3)(-2)]$$

$$= -9[-12 + 12] = 0.$$

$$(c) \frac{d}{dx} g(f(x)) = g'(f(x))f'(x)$$

At $x = -1$, the derivative is

$$g'(f(-1))f'(-1) = g'(0)f'(-1) = (4)(2) = 8.$$

$$(d) \frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

At $x = -1$, the derivative is

$$f'(g(-1))g'(-1) = f'(-1)g'(-1) = (2)(1) = 2.$$

$$(e) \frac{d}{dx} \frac{f(x)}{g(x) + 2} = \frac{(g(x) + 2)f'(x) - f(x)g'(x)}{(g(x) + 2)^2}$$

At $x = 0$, the derivative is

$$\frac{(g(0) + 2)f'(0) - f(0)g'(0)}{(g(0) + 2)^2} = \frac{(-3 + 2)(-2) - (-1)(4)}{(-3 + 2)^2}$$

$$= 6.$$

$$(f) \frac{d}{dx} g(x + f(x)) = g'(x + f(x)) \frac{d}{dx} (x + f(x))$$

$$= g'(x + f(x))(1 + f'(x))$$

At $x = 0$, the derivative is $g'(0 + f(0))(1 + f'(0))$

$$= g'(0 - 1)[1 + (-2)] = (1)(-1) = -1$$

$$68. \frac{dw}{ds} = \frac{dw}{dr} \frac{dr}{ds} = \frac{d}{dr} [\sin(\sqrt{r} - 2)] \frac{d}{ds} \left[8 \sin \left(s + \frac{\pi}{6} \right) \right]$$

$$= \left[\cos(\sqrt{r} - 2) \frac{1}{2\sqrt{r}} \right] \left[8 \cos \left(s + \frac{\pi}{6} \right) \right]$$

At $s = 0$, we have $r = 8 \sin \left(0 + \frac{\pi}{6} \right) = 4$ and so

$$\frac{dw}{ds} = \left[\cos(\sqrt{4} - 2) \frac{1}{2\sqrt{4}} \right] \left[8 \cos \left(0 + \frac{\pi}{6} \right) \right]$$

$$= \left(\frac{\cos 0}{4} \right) \left(8 \cos \frac{\pi}{6} \right) = \left(\frac{1}{4} \right) \left(\frac{8\sqrt{3}}{2} \right) = \sqrt{3}$$

69. Solving $\theta^2 t + \theta = 1$ for t , we have

$$t = \frac{1 - \theta}{\theta^2} = \theta^{-2} - \theta^{-1}, \text{ and we may write:}$$

$$\frac{dr}{d\theta} = \frac{dr}{dt} \frac{dt}{d\theta}$$

$$\frac{d}{d\theta} (\theta^2 + 7)^{1/3} = \frac{dr}{dt} \frac{d}{d\theta} (\theta^{-2} - \theta^{-1})$$

$$\frac{1}{3} (\theta^2 + 7)^{-2/3} (2\theta) = \left(\frac{dr}{dt} \right) (-2\theta^{-3} + \theta^{-2})$$

$$\frac{dr}{dt} = \frac{2\theta(\theta^2 + 7)^{-2/3}}{3(-2\theta^{-3} + \theta^{-2})} = \frac{2\theta^4(\theta^2 + 7)^{-2/3}}{3(\theta - 2)}$$

At $t = 0$, we may solve $\theta^2 t + \theta = 1$ to obtain $\theta = 1$,

$$\text{and so } \frac{dr}{dt} = \frac{2(1)^4(1^2 + 7)^{-2/3}}{3(1 - 2)} = \frac{2(8)^{-2/3}}{-3} = -\frac{1}{6}.$$

70. (a) One possible answer:

$$x(t) = 10 \cos \left(t + \frac{\pi}{4} \right), y(t) = 1$$

$$(b) s(0) = 10 \cos \frac{\pi}{4} = 5\sqrt{2}$$

(c) Farthest left:

$$\text{When } \cos \left(t + \frac{\pi}{4} \right) = -1, \text{ we have } s(t) = -10.$$

Farthest right:

$$\text{When } \cos \left(t + \frac{\pi}{4} \right) = 1, \text{ we have } s(t) = 10.$$

70. Continued

(d) Since $\cos \frac{\pi}{2} = 0$, the particle first reaches the origin at

$$t = \frac{\pi}{4}. \text{ The velocity is given by } v(t) = -10 \sin\left(t + \frac{\pi}{4}\right),$$

so the velocity at $t = \frac{\pi}{4}$ is $-10 \sin \frac{\pi}{2} = -10$, and the speed

at $t = \frac{\pi}{4}$ is $|-10| = 10$. The acceleration is given by

$$a(t) = -10 \cos\left(t + \frac{\pi}{4}\right), \text{ so the acceleration at}$$

$$t = \frac{\pi}{4} \text{ is } -10 \cos \frac{\pi}{2} = 0.$$

71. (a) $\frac{ds}{dt} = \frac{d}{dt}(64t - 16t^2) = 64 - 32t$

$$\frac{d^2s}{dt^2} = \frac{d}{dt}(64 - 32t) = -32$$

(b) The maximum height is reached when $\frac{ds}{dt} = 0$, which occurs at $t = 2$ sec.

(c) When $t = 0$, $\frac{ds}{dt} = 64$, so the velocity is 64 ft/sec.

(d) Since $\frac{ds}{dt} = \frac{d}{dt}(64t - 2.6t^2) = 64 - 5.2t$, the maximum height is reached at $t = \frac{64}{5.2} \approx 12.3$ sec. The maximum height is $s\left(\frac{64}{5.2}\right) \approx 393.8$ ft.

72. (a) Solving $160 = 490t^2$, it takes $\frac{4}{7}$ sec. The average velocity is $\frac{160}{4} = 280$ cm/sec.

(b) Since $v(t) = \frac{ds}{dt} = 980t$, the velocity is $(980)\left(\frac{4}{7}\right) = 560$ cm/sec. Since $a(t) = \frac{dv}{dt} = 980$, the acceleration is 980 cm/sec².

73. $\frac{dV}{dx} = \frac{d}{dx}\left[\pi\left(10 - \frac{x}{3}\right)x^2\right] = \frac{d}{dx}\left[\pi\left(10x^2 - \frac{1}{3}x^3\right)\right]$
 $= \pi(20x - x^2)$

74. (a) $r(x) = \left(3 - \frac{x}{40}\right)^2 x = 9x - \frac{3}{20}x^2 + \frac{1}{1600}x^3$

(b) The marginal revenue is

$$r'(x) = 9 - \frac{3}{10}x + \frac{3}{1600}x^2$$

$$= \frac{3}{1600}(x^2 - 160x + 4800)$$

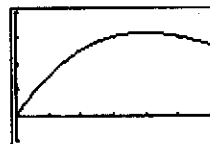
$$= \frac{3}{1600}(x - 40)(x - 120),$$

which is zero when $x = 40$ or $x = 120$. Since the bus holds only 60 people, we require $0 \leq x \leq 60$. The marginal revenue is 0 when there are 40 people, and the

corresponding fare is $p(40) = \left(3 - \frac{40}{40}\right)^2 = \4.00 .

(c) One possible answer:

If the current ridership is less than 40, then the proposed plan may be good. If the current ridership is greater than or equal to 40, then the plan is not a good idea. Look at the graph of $y = r(x)$.



$[0, 60]$ by $[-50, 200]$

75. (a) Since $x = \tan \theta$, we have

$$\frac{dx}{dt} = (\sec^2 \theta) \frac{d\theta}{dt} = -0.6 \sec^2 \theta. \text{ At point A, we have}$$

$$\theta = 0 \text{ and } \frac{dx}{dt} = -0.6 \sec^2 0 = -0.6 \text{ km/sec.}$$

(b) $0.6 \frac{\text{rad}}{\text{sec}} \cdot \frac{1 \text{ revolution}}{2\pi \text{ rad}} \cdot \frac{60 \text{ sec}}{1 \text{ min}} = \frac{18}{\pi}$ revolutions per minute or approximately 5.73 revolutions per minute.

76. Let $f(x) = \sin(x - \sin x)$. Then

$$f'(x) = \cos(x - \sin x) \frac{d}{dx}(x - \sin x)$$

$$= \cos(x - \sin x)(1 - \cos x). \text{ This derivative is zero when}$$

$\cos(x - \sin x) = 0$ (which we need not solve) or

when $\cos x = 1$, which occurs at $x = 2k\pi$ for integers k . For

each of these values, $f(x) = f(2k\pi) = \sin(2k\pi - \sin 2k\pi) = \sin(2k\pi - 0) = 0$. Thus, $f(x) = f'(x) = 0$ for $x = 2k\pi$,

which means that the graph has a horizontal tangent at each of these values of x .